Multiplication of Diagonal Transforms of Loewner Matrices

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ABSTRACT

A new formula for multiplication of matrices arising as appropriate diagonal transformations of Loewner matrices is derived. In the case of symmetric Loewner matrices it is shown that special subclasses form multiplicative groups. Some extensions of known results about polynomials associated with Hankel and Loewner matrices are given. Also, a previously found formula of the author describing the inverses of Loewner matrices is shown to be a consequence of the present results.

INTRODUCTION

Loewner matrices are matrices introduced by K. Loewner in [10]. Loewner studied their connection with the problem of characterizing the so-called monotone matrix functions and also with the problem of rational interpolation. These questions were further investigated in [1], [2], and [3]. Fiedler showed a close connection of the classes of Loewner and Hankel matrices. This provides a good chance for finding various properties of Loewner matrices. On this basis the author has obtained a result concerning the inverse of a nonsingular Loewner matrix [12]. The result of Fiedler and Pták [6] giving a formula for multiplication of Bézout and Hankel matrices led to the idea of finding an analogous formula for Loewner matrices. In this paper we derive such a formula. It is based on the concept of compatibility of a Loewner matrix and a polynomial (which is introduced analogically, as is done for the Hankel case in [4]). The formula concerns the multiplication of matrices which are products of Loewner matrices compatible with a given

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polynomial and of diagonal matrices associated with this polynomial. If we use symmetric Loewner matrices, we obtain classes (each of them compatible with the given polynomial) which have the group structure. In both symmetric and nonsymmetric cases we obtain the mentioned formula for the inverse of a nonsingular Loewner matrix as a corollary.

As a tool we derive theorems which describe the connection between polynomials associated with Loewner matrices and those associated with the corresponding Hankel matrices (this correspondence is due to Fiedler [3]).

0. DEFINITIONS AND PRELIMINARIES

We shall use the following notation:

$r(M)$ for the rank of the matrix M,
$\deg f(x)$ for the degree of the polynomial $f(x)$,
(f, g) for the greatest common divisor of the polynomials $f(x)$ and $g(x)$,
$\operatorname{diag}(d_i)_{i=0}^{n-1}$ for the diagonal matrix with diagonal entries $d_0, \dots, d_{n-1}$.

Throughout the paper all matrices are square of a fixed order n (unless otherwise mentioned).

Definition 0.1. A *Loewner matrix* is a matrix of the form

$$L = \left(\frac{c_i - d_j}{y_i - z_j} \right)_{i,j=0}^{n-1} \tag{0.1}$$

where the c_k, d_k, y_k, z_k, $k = 0, 1, \dots, n-1$, are given complex numbers, all $y_0, \dots, y_{n-1}, z_0, \dots, z_{n-1}$ being mutually distinct.

A *symmetric Loewner matrix* is a matrix of the form

$$L = (l_{ij})_{i,j=0}^{n-1},$$
$$l_{ij} = \frac{c_i - c_j}{y_i - y_j} \qquad \text{for} \quad i \neq j, \tag{0.2}$$

where the c_k, y_k, l_{kk}, $k = 0, \dots, n-1$, are given complex numbers, $y_0, \dots, y_{n-1}$ being mutually distinct.

$\mathscr{L}(y,z)$ will denote the class of all Loewner matrices (0.1) associated with the given so-called *interpolation vectors*

$$y=(y_0,\dots,y_{n-1})^T \quad\text{and}\quad z=(z_0,\dots,z_{n-1})^T$$

formed by $2n$ given distinct complex numbers y_k, z_k.

$\mathscr{L}(y,y)$ will denote the class of all symmetric Loewner matrices (0.2) associated with one interpolation vector

$$y=(y_0,\dots,y_{n-1})^T$$

formed by n distinct numbers y_k.

REMARK 0.1. In the whole paper, in writing $\mathscr{L}(y,z)$ we shall admit $y=z$, that is, all assertions will be valid for both Loewner matrices and symmetric Loewner matrices.

REMARK 0.2. In the third section and in the following discussion we shall need more than two interpolation vectors. That is why we shall use the notation $y^{(k)}$, $k=0,1,\dots,$ instead of y, z.

DEFINITION 0.2. A *Hankel matrix* is a matrix of the form

$$H=(\alpha_{i+j})_{i,j=0}^{n-1}$$

where α_k, $k=0,1,\dots,2n-2$, are given complex numbers. $\mathscr{H}$ will denote the class of all Hankel matrices.

DEFINITION 0.3. A *Bézout matrix* is a matrix

$$B=(b_{ij})_{i,j=0}^{n-1}$$

where

$$\sum_{i,j=0}^{n-1} b_{ij}x^iy^j=\frac{f(x)g(y)-f(y)g(x)}{x-y}$$

for a pair of polynomials $f(x)$, $g(x)$ of degrees at most n. If both $f(x)$, $g(x)$

have degree less than n, the matrix B has the block form

$$B = \begin{pmatrix} \tilde{B} & 0 \\ 0 & 0 \end{pmatrix}.$$

We shall write $B = B(f, g)$.

DEFINITION 0.4. A matrix M is said to have the *H-property* if all minors of order $r(M)$ are different from zero.

DEFINITION 0.5. We introduce the following notation for polynomials associated with interpolation vectors:

$$a(x) = \prod_{j=0}^{n-1} (x - y_j), \qquad a_i(x) = \frac{a(x)}{x - y_i},$$

$$b(x) = \prod_{j=0}^{n-1} (x - z_j), \qquad b_i(x) = \frac{b(x)}{x - z_i},$$

$$a^{(k)}(x) = \prod_{j=0}^{n-1} \left(x - y_j^{(k)}\right), \qquad a_i^{(k)}(x) = \frac{a^{(k)}(x)}{x - y_i^{(k)}}, \qquad k = 0, 1, \dots .$$

1. LOEWNER AND HANKEL MATRICES AND POLYNOMIALS

For our discussion we shall need some notions and assertions which are stated in [3], [4]. We shall extend the concept of compatibility of polynomials and Hankel matrices to compatibility of polynomials and Loewner matrices. In Lemmas 1.2 and 1.3 we recall the known connections between rational functions and Hankel matrices (see e.g. [9]) and Loewner matrices (discovered by Loewner [10]; see also [1], [2], [4]), respectively. Theorem 1.2 and especially Theorems 1.3 and 1.4 describe connections between interpolating rational functions associated with singular Loewner and Hankel matrices corresponding to each other in the isomorphism. If the Loewner or Hankel matrix is nonsingular, then they possess one-parameter families of interpolating functions, and this situation is described in Theorem 1.5 in detail.

DEFINITION 1.1. Let us define the map $\mathscr{J} : \mathscr{H} \to \mathscr{L}(y, z)$ by the equality

$$\mathscr{J}(H) = L \qquad \text{if} \quad L = W_y H W_z^T, \tag{1.1}$$

where

$$W_y = \left(\frac{a_i^{(j)}(0)}{j!} \right)_{i,j=0}^{n-1} \tag{1.2}$$

[the index (j) denotes the jth derivative], and W_z is defined analogously. (It is easy to prove that W_y, W_z are nonsingular.)

THEOREM 1.1 (see [3]). *The map $\mathscr{J}$ is an isomorphism of the classes $\mathscr{H}$ and $\mathscr{L}(y, z)$.*

DEFINITION 1.2 (see [4]). Let H be an arbitrary Hankel matrix (singular or nonsingular) and

$$f(x) = f_n x^n + f_{n-1} x^{n-1} + \cdots + f_0$$

(not necessarily of exact degree n). We shall say H is *compatible* with $f(x)$ if

$$\tilde{H} f = 0,$$

where

$$f = (f_0, \dots, f_n)^T \quad \text{and} \quad \tilde{H} = (\alpha_{i+j})_{i=0}^{n-2}\,{}_{j=0}^{n}$$

[an $(n-1) \times (n+1)$ matrix].

DEFINITION 1.3. Let $L \in \mathscr{L}(y, z)$ be an arbitrary Loewner matrix (singular or nonsingular), and

$$f(x) = f_n x^n + f_{n-1} x^{n-1} + \cdots + f_0$$

(not necessarily of exact degree n). We shall say that L is *compatible* with $f(x)$ if the Hankel matrix H corresponding to L in the isomorphism $\mathscr{J}^{-1}$ (see Definition 1.1) is compatible with $f(x)$.

REMARK 1.1. In the case of nonsymmetric Loewner matrices we could give a definition of compatibility analogous to Definition 1.2. We could divide the set $y_0, \dots, y_{n-1}, z_0, \dots, z_{n-1}$ into two subsets with $n-1$ and

$n+1$ elements and form an $(n-1)\times(n+1)$ Loewner matrix $\tilde{L}$ associated with the same parameters as L, using the first subset for rows and the second for columns. Then we could define the compatibility by

$$\tilde{L}f_{\tilde{L}}=0,$$

where $f_{\tilde{L}}$ are the coefficients of $f(x)$ in a special basis associated with the column subset of interpolation points of $\tilde{L}$.

For instance, if the row subset is $y_0,\dots,y_{n-2}$ and the column subset is $y_{n-1}, z_0,\dots,z_{n-1}$, then

$$\tilde{L}=\begin{pmatrix} \dfrac{c_0-c_{n-1}}{y_0-y_{n-1}} & \dfrac{c_0-d_0}{y_0-z_0} & \cdots & \dfrac{c_0-d_{n-1}}{y_0-z_{n-1}} \\ \dfrac{c_1-c_{n-1}}{y_1-y_{n-1}} & \dfrac{c_1-d_0}{y_1-z_0} & \cdots & \dfrac{c_1-d_{n-1}}{y_1-z_{n-1}} \\ \vdots & \vdots & & \vdots \\ \dfrac{c_{n-2}-c_{n-1}}{y_{n-2}-y_{n-1}} & \dfrac{c_{n-2}-d_0}{y_{n-2}-z_0} & \cdots & \dfrac{c_{n-2}-d_{n-1}}{y_{n-2}-z_{n-1}} \end{pmatrix},$$

$$f_{\tilde{L}}=\left(f_{\tilde{L}}^{(0)},\dots,f_{\tilde{L}}^{(n)}\right)^T,$$

if

$$f(x)=\sum_{i=0}^{n} f_{\tilde{L}}^{(i)}\tilde{a}_i(x),$$

where

$$\tilde{a}_0(x)=\prod_{j=0}^{n-1}(x-\tilde{z}_j),$$

$$\tilde{a}_k(x)=\frac{(x-y_{n-1})\prod_{j=0}^{n-1}(x-\tilde{z}_j)}{x-\tilde{z}_{k-1}}, \qquad k=1,\dots,n.$$

The equivalence of such a definition (and independence of the splitting of the set of interpolation points into rows and columns) is based on the remark following Theorem 12 of [3].

The case of symmetric Loewner matrices would be formally more complicated.

Now we shall recall the connections of Hankel and Loewner matrices and rational functions.

DEFINITION 1.4. Let $H = (\alpha_{i+j})_{i,j=0}^{n-1}$ be an arbitrary Hankel matrix (singular or nonsingular), and $f(x), h(x)$ $[\deg h(x) < \deg f(x) \leqslant n]$ be polynomials (not necessarily relatively prime). We shall say that the rational function $h(x)/f(x)$ is an *interpolating function of* H and write

$$H = H_{h/f}$$

if

$$\frac{h(x)}{f(x)} = \alpha_0 x^{-1} + \alpha_1 x^{-2} + \cdots + \alpha_{2n-2} x^{-(2n-1)} + \alpha_{2n-1} x^{-2n} + \cdots$$

for some $\alpha_{2n-1}, \alpha_{2n}, \dots$ (in some neighborhood of infinity).

(Hankel forms associated with rational functions play an important role in [9].)

LEMMA 1.1. *If* $(h, f) = 1$ *then*

$$r(H_{h/f}) = \deg f(x).$$

(This property is well known. See e.g. [9].)

DEFINITION 1.5. Let $L \in \mathscr{L}(y, z)$ be an arbitrary Loewner matrix (singular or nonsingular), $f(x), q(x)$ polynomials of degrees at most n (not necessarily relatively prime), and $w(x)$ a polynomial of degree at most $2n-1$. We shall say that the rational function $q(x)/f(x)$ is an *interpolating function of* L and write

$$L = L_{q/f}$$

if $q(x)/f(x)$, after deleting the common factors, assumes the values c_i and d_i

respectively at the points y_i and z_i, $i = 0, 1, \ldots, n-1$, and if

$$L = \left(\frac{c_i - d_j}{y_i - z_j} \right)_{i,j=0}^{n-1}.$$

(We admit also symmetric Loewner matrices, demanding in that case that the l_{ii}'s be the first derivatives of $q(x)/f(x)$ with respect to the y_i's.)

Analogously we say that $w(x)$ is an *interpolating polynomial of* L and write

$$L = L_w$$

if

$$w(y_i) = c_i, \quad w(z_i) = d_i, \qquad i = 0, 1, \ldots, n-1,$$

with

$$w'(y_i) = l_{ii}$$

for a symmetric Loewner matrix.

Lemma 1.2. *If* $(q, f) = 1$ *then*

$$r(L_{q/f}) = \max(\deg q(x), \deg f(x)).$$

This assertion is due to Loewner.

Definition 1.6 (see [5]). A singular Hankel matrix $H = (\alpha_{i+j})_{i,j=0}^{n-1}$ is called *proper* if the leading minor

$$(\alpha_{i+j})_{i,j=0}^{r-1}$$

of order $r = r(H)$ is different from zero.

A singular Hankel matrix is called *degenerate* if it has the form

$$H = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}$$

where Z is a square matrix of order less than n.

LEMMA 1.3 (see [5, Theorem 2.5]; in other terms it is contained in [8, Theorem 5.9]). *Every singular Hankel matrix H can be uniquely written in the form*

$$H = H_P + H_D,$$

where H_P is proper and H_D degenerate. Moreover,

$$r(H) = r(H_P) + r(H_D).$$

LEMMA 1.4. *A singular Hankel matrix H has an interpolating rational function $h(x)/f(x)$ if and only if it is proper. The interpolation function is unique, that is, $h(x)$, $f(x)$ such that $(f, h) = 1$ are unique up to a constant multiple.*

The proof follows from [9] and some simple additional considerations.

LEMMA 1.5. *A singular Loewner matrix*

$$L = \left(\frac{c_i - d_j}{y_i - z_j} \right)$$

possesses an interpolating rational function $q(x)/f(x)$ if and only if it has the H-property. The interpolating function is unique for a given fixed choice of the c_i's and d_i's (the matrix L determines the c_i's and d_i's up to an additive constant), so that $f(x)$, $q(x)$ such that $(f, q) = 1$ are unique up to a constant multiple.

(Each Loewner matrix possesses an interpolating polynomial $w(x)$, unique for given c_i's and d_i's.)

This is one of the basic results on Loewner matrices. It is due to Loewner [10]. Various variants or extensions can be found in [1], [2], [3]. The interpolating polynomial will be used for matrices which do not have the H-property. Its existence is evident.

In the following theorems we shall investigate the problem of existence of interpolating functions for a Hankel matrix H and for the Loewner matrix L corresponding to it in the isomorphism $\mathcal{J}$, as well as the connection between these interpolating functions.

THEOREM 1.2. *Let $H \in \mathcal{H}$ and $L \in \mathcal{L}(y, z)$ such that*

$$L = W_y H W_z^T \qquad (i.e. \quad L = \mathcal{J}(H))$$

(see (1.1), (1.2)), *let* $r(H) = r < n$, *and let a fixed choice of the* c_i*'s and* d_i*'s be given such that*

$$L = \left(\frac{c_i - d_j}{y_i - z_j} \right)_{i,j=0}^{n-1}.$$

Then:

(i) *There exist polynomials* $f(x), h(x)$, $\deg h(x) < \deg f(x) < n$, $(f, h) = 1$ *(unique up to a constant multiple) such that*

$$H = H_{h/f} + H_D = H_P + H_D,$$

H_P *proper,* H_D *degenerate,*

$$r(H_P) = \deg f(x), \qquad r(H_D) = r(H) - \deg f(x).$$

(ii) $r(L) = r(H)$.

(iii) *There exist* $\tilde{f}(x), q(x)$, $\deg q(x) < n$, $\deg \tilde{f}(x) < n$, $(\tilde{f}, q) = 1$, *such that*

$$L = L_{q/\tilde{f}}$$

if and only if

$$(f, ab) = 1$$

(here $f(x)$ *is the polynomial from* (i); *for* $a(x), b(x)$ *see Definition* 0.5).

(iv) *If both equivalent properties from* (iii) *hold, then* $\tilde{f}(x), q(x)$ *are unique up to a constant multiple (due to Lemma* 1.5) *and*

$$\tilde{f}(x) = cf(x) \ (c \text{ a constant}). \tag{1.3}$$

Moreover,

$$H \text{ is proper} \quad \text{iff} \quad \deg q(x) = \deg f(x)$$

and

$$H \text{ is degenerate} \quad \text{iff} \quad \deg f(x) = 0.$$

In the general case, the rational function $q(x)/f(x)$ can be uniquely written in the form

$$\frac{q(x)}{f(x)} = \frac{q_0(x)}{f(x)} + \pi(x)$$

for some polynomials $q_0(x), \pi(x)$, $\deg q_0(x) < \deg f(x)$, *and then the corresponding decomposition of L fulfills*

$$L_{q_0/f} = \mathscr{J}(H_P), \qquad L_{\pi/1} = \mathscr{J}(H_D).$$

Proof. (i) follows from Lemma 1.3 and Lemma 1.4; (ii) is evident. (iii) is proved in [13]. The property (1.3) can be proved by means of the concept of H-polynomials (see Fiedler [5]) and analogous L-polynomials (also used by Fiedler). Let us remark that this could also be deduced from Theorem 1.3 which follows. The rest of (iv) is a consequence of the equalities

$$r(H_P) = \deg f(x),$$

$$r(L) = r(H) = \max(\deg q(x), \deg f(x)) \qquad \text{(Lemma 1.2)}. \qquad \blacksquare$$

In the rest of this section we shall derive an equality which might be used for a characterization of compatibility of a Loewner matrix and a polynomial $f(x)$ by means of the interpolating polynomial $w(x)$. For a Loewner matrix having the H-property it may be also considered as a connection between the interpolating functions $q(x)/f(x)$ and $h(x)/f(x)$ in the equality

$$L_{q/f} = W_y(H_{h/f} + H_D)W_z^T.$$

LEMMA 1.6. *Let H be an arbitrary singular Hankel matrix and y, z two vectors each of them having n distinct components. Let W_y and W_z be defined as in Definition* 1.1. *and let*

$$V_y = \left(y_j^i\right)_{i,j=0}^{n-1}, \qquad V_z = \left(z_j^i\right)_{i,j=0}^{n-1}$$

be the corresponding Vandermonde matrices.

Then H can be written as

$$H = H_{h/f} + H_D, \quad \textit{where} \quad (h, f) = 1$$

and there exists a polynomial $q(x)$ *of degree at most n such that*

$$\operatorname{diag}(f(y_k))_{k=0}^{n-1} W_y H W_z^T \operatorname{diag}(f(z_k))_{k=0}^{n-1} = V_y^T B(q, f) V_z. \tag{1.4}$$

The polynomial $q(x)$ *fulfills the equation*

$$q + hab = fw \tag{1.5}$$

for some $w(x)$. *Moreover,*

$$\max(\deg q(x), \deg f(x)) = r(H).$$

For a given fixed $f(x)$ *all the polynomials* $q(x)$ *are of the form*

$$q_0(x) + cf(x).$$

Proof.

(1) We shall first prove the lemma for certain simple types of Hankel matrices. Suppose that

$$H = H_{1/(x-t)^\nu}. \tag{1.6}$$

In all the calculations let us write

$$A(x) = a(x)b(x).$$

Writing the expansion of $A(x)$ at the point $x = t$, we easily obtain

$$-\sum_{p=0}^{\nu-1} \frac{A^{(p)}(t)}{p!}(x-t)^p + 1 \cdot A(x) = (x-t)^\nu p(x)$$

for some $p(x)$. We shall prove that

$$-\sum_{p=0}^{\nu-1} \frac{A^{(p)}(t)}{p!}(x-t)^p \tag{1.7}$$

is the required $q(x)$. Actually, for this $q(x)$ the ij-element on the right-hand side of (1.4) is (due to Definition 0.3)

$$\left(V_y^T B \left(-\sum_{p=0}^{\nu-1} \frac{A^{(p)}(t)}{p!}(x-t)^p, (x-t)^\nu \right) V_z \right)_{ij}$$

$$= \sum_{p=0}^{\nu-1} \sum_{l=0}^{\nu-1-p} (y_i - t)^{\nu-p-1}(z_j - t)^{p+l} \frac{A^{(p)}(t)}{p!}. \tag{1.8}$$

Expressing $A^{(p)}(t)/p!$ as

$$\frac{(t-y_i)(t-z_j)\left[a_i(x)b_j(x)\right]_{x=t}^{(p)}}{p!} + \frac{(t-y_i)\left[a_i(x)b_j(x)\right]_{x=t}^{(p-1)}}{(p-1)!}$$

$$+ \frac{(t-z_j)\left[a_i(x)b_j(x)\right]_{x=t}^{(p-1)}}{(p-1)!} + \frac{\left[a_i(x)b_j(x)\right]_{x=t}^{(p-2)}}{(p-2)!}$$

we obtain after simplification that the expression in (1.8) equals

$$(y_i - t)^\nu (z_j - t)^\nu \left[a_i(x)b_j(x)\right]_{x=t}^{(\nu-1)} \Big/ (\nu-1)!$$

$$= (y_i - t)^\nu (z_j - t)^\nu \sum_{k=0}^{n-1} \frac{a_i^{(k)}(0)}{k!} \frac{b_j^{(l)}(0)}{l!} \binom{k+l}{\nu-1} t^{k+l+1-\nu}$$

$$= \left(\operatorname{diag}(f(y_k))\, W_y H_{1/(x-t)^\nu} W_z^T \operatorname{diag}(f(z_k))\right)_{ij},$$

since

$$H_{1/(x-t)^\nu} = (\alpha_{k+l})_{k,p=0}^{n-1},$$

where

$$\alpha_s = \binom{s}{\nu-1} t^{s+1-\nu}.$$

Further suppose that H is degenerate,

$$H = (\alpha_{i+j})_{i,j=0}^{n-1}, \qquad \alpha_k = \delta_{k,N-1}, \quad n < N \leqslant 2n-1. \tag{1.9}$$

By an analogous calculation to the preceeding case we can show that the matrix $B(q,1)$,

$$q(x)=\sum_{p=1}^{\nu}\frac{A^{(N+p)}(0)}{(N+p)!}x^{p},\qquad \nu=2n-N, \tag{1.10}$$

fulfills (1.4). Equation (1.5) has the trivial form

$$q=0\cdot ab=1\cdot q.$$

[$q(x)$ could have an arbitrary absolute term and so we set it equal to zero.] Note that

$$\deg q(x)=r(H).$$

(2) In the general case we decompose the Hankel matrix into the simple matrices of the forms (1.6) and (1.9): We can write

$$f(x)=\prod_{i=1}^{s}(x-t_i)^{\nu_i}$$

and then

$$H=H_{h/f}+H_D=\sum_{i=1}^{s}\sum_{j=1}^{\nu_i}H_{h_{ij}/f_{ij}}+H_D,$$

where

$$f_{ij}(x)=(x-t_i)^j,\qquad h_{ij}\ \text{constants}.$$

We know that

$$\operatorname{diag}\big(f_{ij}(y_k)\big)W_yH_{h_{ij}/f_{ij}}W_z^T\operatorname{diag}\big(f_{ij}(z_k)\big)=V_y^TB(q_{ij},f_{ij})V_z, \tag{1.11}$$

$$q_{ij}+h_{ij}ab=fw_{ij}, \tag{1.12}$$

and [summing the expressions for degenerate matrices of the form (1.9)]

$$W_yH_DW_z^T=V_y^TB(q_\infty,1)V_z, \tag{1.13}$$

where $q_\infty(x)$ is some polynomial of degree $r(H_D)$. From (1.11) we can obtain

$$\operatorname{diag}(f(y_k))W_y H_{h_{ij}/f_{ij}} W_z^T \operatorname{diag}(f(z_k)) = V_y^T B(q_{ij}f/f_{ij}, f)V_z, \quad (1.14)$$

and from (1.13)

$$\operatorname{diag}(f(y_k))W_y H_D W_z^T \operatorname{diag}(f(z_k)) = V_y^T B(q, f)V_z. \qquad (1.15)$$

Summing (1.14) for $i = 1, \dots, s$, $j = 1, \dots, \nu_i$ gives, with (1.15),

$$\operatorname{diag}(f(y_k))W_y(H_{h/f} + H_D)W_z^T \operatorname{diag}(f(z_k)) = V_y^T B(q, f)V_z,$$

where

$$\begin{aligned} q &= \sum_{i=1}^{s} \sum_{j=1}^{\nu_i} \frac{q_{ij}f}{f_{ij}} + q_\infty f, \\ h &= \sum_{i=1}^{s} \sum_{j=1}^{\nu_i} \frac{h_{ij}f}{f_{ij}}, \end{aligned} \qquad (1.16)$$

and obviously

$$q + hab = fw$$

if

$$w = \sum_{i=1}^{s} \sum_{j=1}^{\nu_i} w_{ij} + q_\infty.$$

In the case

$$\deg q(x) > \deg f(x)$$

the degree of $q(x)$ is [see (1.16)]

$$\deg q(x) = \deg q_\infty(x) + \deg f(x) = r(H_D) + r(H_P) = r(H).$$

In the case

$$\deg q(x) \leqslant \deg f(x),$$

the degree of $q_\infty(x)$ is zero, so that

$$H_D = 0$$

and

$$\deg f(x) = r(H_p) = r(H).$$

In both cases we can write

$$r(H) = \max(\deg q(x), \deg f(x)).$$

(3) If two polynomials $q(x)$ and $\bar{q}(x)$ fulfill (1.4), then

$$B(q, f) = B(\bar{q}, f),$$

and it is easy to show that

$$\bar{q}(x) = q(x) + cf(x). \qquad \blacksquare$$

THEOREM 1.3. *Let $H \in \mathscr{H}$ and $L \in \mathscr{L}(y, z)$ be given such that*

$$L = W_y H W_z^T = \mathscr{J}(H) \qquad (\textit{see } (1.1), (1.2)).$$

Then H can be written as

$$H = H_{h/f} + H_D, \qquad (h, f) = 1$$

(for a unique $f(x)$ up to a constant multiple) and L as

$$L = L_w.$$

For given fixed $w(x)$, $f(x)$ and the corresponding $h(x)$ there is a unique polynomial $q(x)$ of degree at most n such that

$$q + hab = fw. \tag{1.17}$$

Moreover,

$$r(L) = \max(\deg q(x), \deg f(x)),$$

and in the case $(f, ab) = 1$,

$$L = L_{q/f}.$$

Proof. Let us take the polynomial $q(x)$ from the preceding lemma. Then

$$\max(\deg q(x), \deg f(x)) = r(H) = r(L).$$

(1) If $(f, ab) = 1$, the diagonals in (1.4) are nonsingular, so that

$$L = [\operatorname{diag}(f(y_k))]^{-1} V_y^T B(q, f) V_z [\operatorname{diag}(f(z_k))]^{-1}.$$

But the right-hand side is just $L_{q/f}$. From the equality

$$\frac{q}{f} + \frac{h}{f} ab = w$$

it directly follows that

$$L_{q/f} = L_w.$$

(2) In the case $(f, ab) \neq$ const we can take interpolation vectors $y_\varepsilon, z_\varepsilon$ with distinct components as functions of ε in some neighborhood of zero such that

$$\lim_{\varepsilon \to 0} y_\varepsilon = y, \qquad \lim_{\varepsilon \to 0} z_\varepsilon = z,$$

and for the corresponding $a_\varepsilon(x), b_\varepsilon(x)$

$$(f, a_\varepsilon b_\varepsilon) = 1 \text{ for } \varepsilon \neq 0.$$

Then we can construct the corresponding $W_{y_\varepsilon}, W_{z_\varepsilon}$ and

$$L_\varepsilon = W_{y_\varepsilon} H_{h/f} W_{z_\varepsilon}^T.$$

From Lemma 1.6 we know that there are $q_\varepsilon(x), w_\varepsilon(x)$ for all $\varepsilon \neq 0$ and $q_0(x), w_0(x)$ such that

$$q_\varepsilon + h a_\varepsilon b_\varepsilon = f w_\varepsilon,$$

$$q_0 + hab = f w_0$$

and

$$L_\varepsilon = L_{q_\varepsilon/f} = L_{w_\varepsilon}.$$

Moreover, the proof of that lemma gives an explicit form of $q_\varepsilon(x)$ [see (1.7) and (1.10)] in dependence on $a_\varepsilon b_\varepsilon$, from which evidently $q_\varepsilon(x)$ has some limit $\tilde{q}_0(x)$. Furthermore, $\deg q_\varepsilon(x) < \deg f(x)$, $\deg q_0(x) < \deg f(x)$. Then $w_\varepsilon(x)$ must also have a limit $\tilde{w}_0(x)$,

$$\tilde{q}_0 + hab = f\tilde{w}_0.$$

From

$$q_0 + hab = fw_0 \quad \text{and} \quad \tilde{q}_0 + hab = f\tilde{w}_0,$$

taking into account the degrees of $q_0(x)$, $\tilde{q}_0(x)$, and $f(x)$, it follows that

$$\tilde{w}_0(x) = w_0(x) \quad \text{and} \quad \tilde{q}_0(x) = q_0(x).$$

Now

$$W_y H_{h/f} W_z^T = \lim_{\varepsilon \to 0} L_\varepsilon = \lim_{\varepsilon \to 0} L_{w_\varepsilon} = L_{\tilde{w}_0} = L_{w_0}.$$

If $q_\infty(x)$ corresponds to H_D as in the proof of Lemma 1.4, then

$$L = L_w,$$

$$q + hab = fw,$$

where

$$w(x) = w_0(x) + q_\infty(x), \qquad q(x) = q_0(x) + q_\infty(x) f(x).$$

(3) The uniqueness of $q(x)$ follows directly from (1.17), since for fixed $f(x)$ the polynomial $h(x)$ is unique. ■

The following lemma shows the connection between compatible polynomials and the polynomial $f(x)$ from the decomposition

$$H = H_{h/f} + H_D.$$

LEMMA 1.7. *Let H be a singular Hankel matrix and $F(x)$ a polynomial of degree at most n. Due to Lemmas* 1.3 *and* 1.4 *we can write*

$$H = H_{h/f} + H_D, \qquad (h, f) = 1,$$

for a polynomial $f(x)$ unique up to a constant multiple. The polynomial $F(x)$ is compatible with H if and only if it is completely divisible by $f(x)$. Here $F(x)$ and $f(x)$ are considered as polynomials of degrees n and r, respectively, and the complete divisibility means the usual divisibility including the property

$$n - \deg F(x) \geqslant r - \deg f(x).$$

(That is, the degree of the root ∞ in $f(x)$ is less than or equal to that in $F(x)$.)

This assertion follows from [4, Theorem 2.2(v)] because $f(x)$ is identical with the H-polynomial [4, 5].

Lemma 1.5 enables us to extend the equality (1.17) to arbitrary compatible polynomials $f(x)$:

THEOREM 1.4. *Let $L \in \mathscr{L}(y, z)$ be an arbitrary singular Loewner matrix, let $f(x)$, $\deg f(x) \leqslant n$, be some polynomial compatible with L, and let $w(x)$, $\deg w(x) \leqslant 2n - 1$, be an interpolating polynomial of L. Then there are unique (for a fixed $w(x)$) polynomials $q(x)$, $h(x)$, $\deg q(x) \leqslant n$, $\deg h(x) < n$, such that*

$$q + hab = fw. \tag{1.18}$$

Moreover, the matrix L can be written as

$$L = W_y\left(H_{h/f} + H_D\right)W_z^T$$

for some degenerate H_D. In the case $(f/(h, f), ab) = 1$, L can be written as

$$L = L_{q/f}$$

(the rational functions $h(x)/f(x)$, $q(x)/f(x)$ are not in the coprime form in general).

Proof. The polynomial $f(x)$ can be written as

$$f(x) = d(x)f_0(x)$$

where

$$L = W_y\left(H_{h_0/f_0} + H_D\right)W_z^T$$

for some degenerate H_D, and

$$(h_0, f_0) = 1, \qquad \deg d(x) \leqslant n - r\left(H_{h_0/f_0} + H_D\right) = n - r(L).$$

From Theorem 1.3

$$q_0 + h_0 ab = f_0 w,$$

so that

$$q_0 d + h_0 dab = fw.$$

Note that $\deg q_0 d \leqslant n$, $\deg h_0 d < n$. If we write $q = q_0 d,\ h = h_0 d$, we obtain (1.18). From this equation it is evident that

$$L = L_{q/f}$$

in the case when $(f/(h, f), ab) = 1$. ∎

REMARK 1.2. It would be possible to use the equation (1.18) as a characterization of compatibility of $f(x)$ and L by means of the interpolating polynomial $w(x)$.

In the case of nonsingular Hankel and Loewner matrices the situation is completely different. We shall characterize compatible polynomials and interpolation functions and describe their connections in the following theorem:

THEOREM 1.5. *Let $H \in \mathscr{H}$ and $L \in \mathscr{L}(y, z)$ be arbitrary nonsingular matrices such that*

$$L = W_y H W_z^T \qquad (i.e.\ L = \mathscr{I}(H)).$$

Let

$$L = L_w.$$

Then there exist polynomials $f(x), h(x), q(x)$ *such that* $\deg f(x) = n$, $\deg h(x) < n$, $\deg q(x) \leqslant n$, *and*

$$q + hab = fw, \tag{1.19}$$

$$H = H_{h/f}, \qquad L = L_{q/f}.$$

Any such $f(x)$ *is relatively prime with* $a(x)b(x)$, *and the system of polynomial equations*

$$fp + gh = 1, \tag{1.20}$$

$$fz - gq = ab \tag{1.21}$$

has a unique solution $p(x)$, $g(x)$, *and* $z(x)$ *satisfying the condition* $\deg g(x) < \deg f(x)$. *In addition, the polynomials* $p(x)$ *and* $z(x)$ *fulfil* $\deg p(x) < \deg f(x)$, $\deg z(x) = n$.

A polynomial $\hat{f}(x)$ *of degree at most n is compatible with H and with L if and only if*

$$\hat{f}(x) = \alpha f(x) + \beta g(x) \tag{1.22}$$

for some constants α, β.

A function $\hat{h}(x)/\hat{f}(x)$ *is an interpolating function for H if and only if*

$$\hat{f}(x) = \alpha f(x) + \beta g(x),$$

$$\hat{h}(x) = \alpha h(x) - \beta p(x)$$

for some α, β, $\alpha \neq 0$.

A function $\hat{q}(x)/\hat{f}(x)$ *is an interpolating function for L if and only if*

$$\hat{f}(x) = \alpha f(x) + \beta g(x),$$

$$\hat{q}(x) = \alpha q(x) + \beta z(x)$$

for some α, β *such that* $(\hat{f}, ab) = 1$.

For arbitrary fixed α, β *the polynomials* $\hat{f}(x) = \alpha f(x) + \beta g(x)$, $\hat{q}(x) = \alpha q(x) + \beta z(x)$, *and* $\hat{h}(x) = \alpha h(x) - \beta p(x)$ *fulfil the equality*

$$\hat{q} + \hat{h}ab = \hat{f}w. \tag{1.23}$$

Moreover, if

$$\hat{f}(x) = \alpha f(x) + \beta g(x)$$

is fixed, then the polynomials $\hat{q}(x) = \alpha q(x) + \beta z(x)$ *and* $\hat{h}(x) = \alpha h(x) - \beta p(x)$ *are the unique polynomials satisfying* (1.23).

Proof. It is well known that all nonsingular Hankel and Loewner matrices possess interpolating functions. The parametrization of these functions for nonsingular Hankel matrices is given by Fuhrmann in [7]. It is also possible to derive it for both Hankel and Loewner matrices from their connections with Bézout matrices (this way is used in [13]). The equality (1.19) can be derived in the same way as in the singular case. The equality (1.23) for general polynomials follows from (1.19) and (1.20), (1.21). It is not difficult to prove that both $H_{\hat{h}/\hat{f}}$ and $L_{\hat{q}/\hat{f}}$ are compatible with $\hat{f}(x)$ for all admissible denominators $\hat{f}(x)$. Since the space of all compatible polynomials is a linear space of dimension two (this follows directly from Definition 1.2), the polynomial $\hat{f}(x)$ from (1.22) is compatible with H and L for all values of α and β and there are no other compatible polynomials. ∎

2. *LD*-MATRICES AND ASSOCIATED POLYNOMIAL OPERATORS

Now we shall define a class of matrices which arise by appropriate diagonal transformations from Loewner matrices and which we shall call *LD*-matrices. We shall base the definition on the concept of compatible polynomials. Our main interest is to study products of *LD*-matrices.

DEFINITION 2.1. A matrix M is an *LD-matrix* if it is of the form

$$M = LD$$

where L is a Loewner matrix of some class $\mathscr{L}(y, z)$ and

$$D = \operatorname{diag}\left(\frac{f(z_i)}{b_i(z_i)} \right)_{i=0}^{n-1}, \tag{2.1}$$

where $f(x)$ is some nonzero polynomial compatible with L. If the polynomial $f(x)$ (not identically zero) and the vectors y and z are given we shall denote by

$$\mathscr{L}\mathscr{d}_f(y, z)$$

the class of all LD-matrices $M = LD$ such that L is compatible with $f(x)$ and D has the form (2.1). Further we shall denote by

$$\mathscr{L}\mathscr{d}_f^{(H)}(y, z)$$

the subclass of all matrices in $\mathscr{L}\mathscr{d}_f(y, z)$ with L having the H-property.

Now we shall describe LD-matrices as representations of some operators on a polynomial space. This will yield a means for the description of products of LD-matrices.

THEOREM 2.1. *Let $M \in \mathscr{L}\mathscr{d}_f^{(H)}(y, z)$, suppose $(f, ab) = 1$ and let*

$$M = L_{q/f}D$$

(see Definition 1.5). Let us write for arbitrary two vectors

$$r = (r_0, r_1, \dots, r_{n-1})^T \quad \textit{and} \quad s = (s_0, s_1, \dots, s_{n-1})^T,$$

$$r(x) = \sum_{i=0}^{n-1} r_i a_i(x),$$

$$s(x) = \sum_{i=0}^{n-1} s_i b_i(x).$$

Then

$$r^T M = s^T \quad \textit{iff} \quad sa = qr + f\pi \tag{2.2}$$

for some polynomial $\pi(x)$ of degree less than n.

REMARK 2.1. The polynomials $q(x)$, $f(x)$, and $a(x)$ are given. It is not difficult to show that for every $r(x)$ there exists a unique pair $\pi(x)$, $s(x)$ such that (2.2) is fulfilled. It can be constructed in the following way: Since

$$(f, a) = 1,$$

there are polynomials $\pi_0(x)$, $s_0(x)$ such that

$$s_0 a = 1 + f\pi_0.$$

Then

$$\tilde{s}a = qr + f\tilde{\pi},$$

where

$$\tilde{s} = s_0 qr,$$

$$\tilde{\pi} = \pi_0 qr.$$

If for some other polyomials $s(x), \pi(x)$ the equation

$$sa = qr + f\pi$$

is fulfilled, then

$$\tilde{s} - s = \nu f,$$

$$\tilde{\pi} - \pi = -\nu a$$

for some polynomial $\nu(x)$. Evidently we can find $\nu(x)$ such that

$$\deg \pi(x) < \deg a(x) = n.$$

Equation (2.2) can also be regarded as equation for interpolating the function

$$\frac{q(x)r(x)}{a(x)}$$

by a polynomial $s(x)$ in the roots (including multiplicities) of $f(x)$. The condition $\deg \pi(x) < n$ means we interpolate also the values at infinity if it is a "root" of $f(x)$. In other words, if $\deg f(x) < n$, the functions $q(x)r(x)$ and $s(x)a(x)$ must have the same coefficients at the highest powers of x.

Proof. Let (2.2) be fulfilled. Then $\pi(x)$ is that polynomial of degree less than n for which

$$\pi(y_i) = -\frac{q(y_i)r(y_i)}{f(y_i)}.$$

Consider first the nonsymmetric case: Let us denote by

$$(L_{q/f})_j$$

the jth column of $L_{q/f}$, and let us multiply out

$$\begin{aligned} r^T(L_{q/f})_j &= \sum_{i=0}^{n-1} \frac{r(y_i)}{a_i(y_i)} \cdot \frac{q(y_i)/f(y_i) - q(z_j)/f(z_j)}{y_i - z_j} \\ &= \sum_{i=0}^{n-1} \frac{\pi(y_i)}{(z_j - y_i)a_i(y_i)} + \frac{q(z_j)}{f(z_j)} \sum_{i=0}^{n-1} \frac{r(y_i)}{(z_j - y_i)a_i(y_i)} \\ &= \frac{\pi(z_j)}{a(z_j)} + \frac{q(z_j)r(z_j)}{f(z_j)a(z_j)} = \frac{s(z_j)}{f(z_j)}. \end{aligned}$$

Now it is already evident that

$$r^T L_{q/f} D = s^T.$$

In the symmetric case ($y = z$) let us write

$$z_{i\varepsilon} = y_i + \varepsilon$$

and assign the matrices

$$L_{q/f}(\varepsilon) \in \mathscr{L}(y, z_\varepsilon), D(\varepsilon)$$

and the vector $s(\varepsilon)$. From the form of these objects we can see that

$$\begin{aligned} &\lim_{\varepsilon \to 0} L_{q/f}(\varepsilon) = L_{q/f} \in \mathscr{L}(y, y), \\ &\lim_{\varepsilon \to 0} D(\varepsilon) = D, \\ &\lim_{\varepsilon \to 0} s(\varepsilon) = s. \end{aligned}$$

Since

$$r^T L_{q/f}(\varepsilon) D(\varepsilon) = s(\varepsilon)^T$$

for all ε in some neighborhood of zero $(0 < |\varepsilon| < \delta)$, we also have

$$r^T L_{q/f} D = s^T.$$

Since both the equations

$$r^T M = s^T \quad \text{and} \quad sa = qr + f\pi$$

determine for every given vector r a unique vector s, the converse is also true. ■

REMARK 2.2. We can ask whether the assumption $(f, ab) = 1$ is necessary in Theorem 2.1.

If L is singular, then any compatible polynomial has the form

$$f(x) = d(x) f_0(x),$$

where

$$L = W_y \left(H_{h_0/f_0} + H_D \right) W_z^T$$

and

$$(h_0, f_0) = 1.$$

The polynomial $d(x)$ can possess common factors with the polynomial $a(x)b(x)$. In this case

$$L = L_{q_0/f_0} = L_{q_0 d/f_0 d}$$

for some $q_0(x)$. If

$$s_0 a = q_0 r + f_0 \pi$$

then

$$r^T M_0 = s_0^T, \qquad \text{where} \quad M_0 = L_{q_0/f_0} \operatorname{diag}\left(\frac{f_0(z_i)}{b_i(z_i)} \right).$$

If

$$M = L_{q/f}\,\mathrm{diag}\left(\frac{f(z_i)}{b_i(z_i)}\right) \quad \text{and} \quad r^T M = s^T$$

then

$$s_i = d(z_i)(s_0)_i$$

so that

$$s(x) = d(x)s_0(x)$$

and

$$sa = qr + f\pi.$$

However, this equation does not determine $s(x)$ uniquely.

If L is nonsingular, then the family of compatible polynomials contains polynomials which have common factors with $a(x)b(x)$. In such a case, neither

$$L = L_{q/f}$$

nor

$$sa = qr + f\pi$$

will hold in general.

EXAMPLE 2.1. Let $L = L_{(-16x+12)/x^2} \in \mathscr{L}(y, z)$, where $y = (1,3)^T$, $z = (2, -2)^T$. Then $a(x) = x(x-1)$, $b(x) = (x-2)(x+2)$, and we can compute that

$$L = \begin{pmatrix} -1 & -5 \\ 1 & -3 \end{pmatrix},$$

a nonsingular matrix.

We can find that the polynomials $z(x)$, $g(x)$ which occur in Theorem 1.5 are $z(x) = x^2 - 4x - 1$, $g(x) = 1$, so that L can be written as

$$L = L_{[\alpha(-16x+12)+\beta(x^2-4x-1)]/(\alpha x^2+\beta)}$$

for all α, β such that $(\alpha x^2+\beta, a(x)b(x)) = (\alpha x^2+\beta, x(x-1)(x-2)(x+2)) = 1$. If we take $\alpha = 1$, $\beta = -1$, then $(\alpha x^2+\beta, a(x)b(x)) = x-1$ and the matrix

$$L_{[1\cdot(-16x+12)-1\cdot(x^2-4x-1)]/(x^2-1)} = L_{(-x^2-12x+12)/(x^2-1)}$$
$$= L_{(-x-13)/(x+1)} = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}$$

differs from L in the first row, since the function $(-x-13)/(x+1)$ has not the prescribed value of the associated interpolation problem at the point 1.

The LD-matrix corresponding to L and $f(x) = x^2-1$ is

$$M = L\,\mathrm{diag}\left(\tfrac{3}{4}, -\tfrac{3}{4}\right) = \frac{3}{4}\begin{pmatrix} -1 & 5 \\ 1 & 3 \end{pmatrix}.$$

If $r = (1, -1)^T$ then

$$r^T M = \left(-\tfrac{3}{2}, \tfrac{3}{2}\right) = s^T.$$

The corresponding polynomials are

$$r(x) \equiv 1, \qquad s(x) \equiv -6,$$

and it is not difficult to show that the equality

$$sa = qr + f\pi$$

for

$$q(x) = -x^2 - 12x + 13, \qquad f(x) = x^2 - 1$$

cannot be fulfilled for any polynomial $\pi(x)$.

REMARK 2.3. In the general case, if

$$M \in \mathscr{L}\mathscr{d}_f(y, z)$$

(that is, if the corresponding matrix L does not necessarily have the H-property), it is possible to describe the corresponding operator by means of the interpolating polynomial $w(x)$: If

$$M = L_w D,$$

then (with the same notation as in Theorem 2.1)

$$r^T M = s^T$$

if and only if

$$aS = wr + \pi, \tag{2.3}$$

$$s = Sf + bW \tag{2.4}$$

for some polynomials $S(x)$, $\pi(x)$, and $W(x)$ with $\deg \pi(x) < n$. [The uniqueness of $s(x)$ follows from the uniqueness of the quotient and remainder in the division of $w(x)r(x)$ by $a(x)$ and of $S(x)f(x)$ by $b(x)$.]

Let us mention that in the case $r(L) < n$ the equality

$$sa = qr + f\pi$$

[where $q(x)$ is the polynmial existing owing to Theorem 1.4] holds for arbitrary $M = LD \in \mathscr{L}\!\!\mathscr{d}_f(y, z)$. However, this equation does not determine the operator corresponding to the matrix M uniquely if $(f, ab) \neq \text{const}$.

REMARK 2.4. In the case $(f, ab) = 1$, $M = L_{q/f} D \in \mathscr{L}\!\!\mathscr{d}_f^{(H)}(y, z)$, and $r(M) < n$, it is easy to describe the range and the kernel of the associated operator (and, equivalently, the kernel of the matrix $L_{q/f}^T$). It is convenient for this purpose to introduce a (homogeneous) form in two variables $\overline{w}(x_1, x_2)$ associated with a given number ρ and an arbitrary polynomial $w(x)$, if it is considered as having the degree ρ [not smaller than the actual degree of $w(x)$]:

$$\overline{w}(x_1, x_2) = x_2^\rho w(x_1/x_2).$$

Suppose that $(q, f) = 1$. The equation (2.2) can be rewritten as

$$\bar{s}\bar{a} = \bar{q}\bar{r} + \bar{f}\bar{\pi}, \tag{2.2$'$}$$

where $a(x)$, $q(x)$, and $f(x)$ are considered as polynomials of degree n, $s(x)$,

$r(x)$, and $\pi(x)$ as polynomials of degree $n-1$. We shall use the concept of the greatest common divisor of forms in the evident sense. [The greatest common divisor $(\bar{q}, \bar{f})$ is a multiple of $x_2^{n-n_1}$ if $\max(\deg q(x), \deg f(x)) = n_1 < n$.] Now from the equivalence of $r^T M = s^T$ and (2.2′) we can obtain easily

$$r^T M = 0^T \quad \text{iff} \quad \bar{r} \text{ is a multiple of } \frac{\bar{f}}{(\bar{q}, \bar{f})}$$

and

$$s^T = r^T M \quad \text{for some } r \quad \text{iff} \quad \bar{s} \text{ is a multiple of } (\bar{q}, \bar{f}).$$

Evidently $r^T M = 0^T$ is equivalent to $r^T L_{q/f} = 0^T$. After generalization to arbitrary Loewner matrices we obtain the following assertion: Let L be an arbitrary singular Loewner matrix in $\mathscr{L}(y, z)$. Then

$$L = W_y (H_{h/f} + H_D) W_z^T, \qquad (h, f) = 1$$

for some $h(x)$, $f(x)$ and

$$r^T L = 0^T$$

if and only if the polynomial $r(x)$ of degree at most $n-1$, defined by

$$r(x) = \sum_{i=0}^{n-1} r_i a_i(x),$$

is completely divisible by $f(x)$ [in the sense that $r(x)$ is considered as a polynomial of degree $n-1$ and $f(x)$ of degree $r(L)$]. (Compare this property with the property of compatibility with L.)

REMARK 2.5. It is possible to show that in each class $\mathscr{L}\!\mathscr{A}_f(y, z)$ there is a matrix representing the identity operator on the space of polynomials. In the symmetric case (and assuming $(f, a) = 1$) this matrix is the identity matrix.

Indeed, if we take

$$q(x) = a(x),$$

then the equation (2.2) has the unique solution

$$s(x)=r(x), \qquad \pi(x)=0$$

(the zero polynomial). That means that

$$r^T L_{a/f} D = s^T \quad \text{iff} \quad s(x)=r(x),$$

and in the case $L_{a/f}D \in \mathscr{L}\!\mathscr{d}_f(y,y)$, $(f,a)=1$ we have

$$L_{a/f}D = I.$$

3. PRODUCTS OF *LD*-MATRICES

We shall describe now some special products of *LD*-matrices. In particular, we shall show that the product of k *LD*-matrices $M_1, M_2, \dots, M_k$, where

$$M_j \in \mathscr{L}\!\mathscr{d}_f\left(y^{(j-1)}, y^{(j)}\right),$$

is again an *LD*-matrix, which belongs to $\mathscr{L}\!\mathscr{d}_f(y^{(0)}, y^{(k)})$ (Theorem 3.2). If

$$M_j \in \mathscr{L}\!\mathscr{d}_f^{(H)}\left(y^{(j-1)}, y^{(j)}\right)$$

for all j, the resulting matrix is again in $\mathscr{L}\!\mathscr{d}_f^{(H)}(y^{(0)}, y^{(k)})$. We can describe the connection of the "interpolating functions" $q_j(x)/f(x)$ and $q(x)/f(x)$ of the Loewner matrices forming M_j and M in this case (Theorem 3.1). In the general case we need for the description of the interpolating polynomials $w_j(x)$ and $w(x)$ both $q_j(x), q(x)$ and $h_j(x), h(x)$ (the polynomials assigned to Loewner matrices in Theorem 1.4).

THEOREM 3.1. *Let $f(x), q_1(x), q_2(x), \dots, q_k(x)$, $k \geqslant 1$, be polynomials of degrees at most n, $f(x)$ different from the zero polynomial. Further let $y^{(0)}, y^{(1)}, \dots, y^{(k)}$ be vectors (each of them having n distinct components) such that*

$$\left(f, a^{(j)}\right)=1, \qquad j=0,1,\dots,k$$

(*the notation has been introduced in Definition* 0.5). *Let us take the matrices*

$$L_{q_j/f} \qquad \text{in } \mathscr{L}\left(y^{(j-1)}, y^{(j)}\right)$$

and define

$$D_j = \operatorname{diag}\left(\frac{f\left(y_i^{(j)}\right)}{a_i^{(j)}\left(y_i^{(j)}\right)} \right)_{i=0}^{n-1}, \tag{3.1}$$

$j = 1, 2, \ldots, k$. *Then*

$$L_{q_1/f} D_1 L_{q_2/f} D_2 \cdots L_{q_k/f} D_k = L_{q/f} D_k \in \mathscr{L}_f^{(H)}\left(y^{(0)}, y^{(k)}\right), \tag{3.2}$$

where $q(x)$ *is a polynomial of degree less than or equal to* n *for which*

$$q a^{(1)} a^{(2)} \cdots a^{(k-1)} = q_1 q_2 \cdots q_k + f W \tag{3.3}$$

for some polynomial $W(x)$ *of degree at most* $(k-1)n$. (*The polynomial* $q(x)$ *is determined uniquely up to a constant multiple of* $f(x)$. *It can be constructed analogously as the pair* $\pi(x), w(x)$ *in Remark* 2.1.)

Proof. Let $k = 2$. By Theorem 2.1,

$$r^T L_{q_1/f} D_1 L_{q_2/f} D_2 = t^T$$

if and only if both

$$s a^{(0)} = q_1 r + f \pi_1, \qquad \deg \pi_1(x) < n,$$

and

$$t a^{(1)} = q_2 s + f \pi_2, \qquad \deg \pi_2(x) < n,$$

are fulfilled, where

$$s^T = r^T L_{q_1/f} D_1,$$

$$r(x) = \sum_{i=0}^{n-1} a_i^{(0)}(x) r_i, \qquad s(x) = \sum_{i=0}^{n-1} a_i^{(1)}(x) s_i,$$

$$t(x) = \sum_{i=0}^{n-1} a_i^{(2)}(x) t_i.$$

Then

$$t a^{(1)} a^{(0)} = q_1 q_2 r + q_2 f \pi_1 + a^{(0)} f \pi_2.$$

Defining $q(x)$ by

$$q a^{(1)} = q_1 q_2 + f W, \qquad \deg W \leqslant n,$$

we obtain

$$\begin{aligned} t a^{(1)} a^{(0)} &= a q^{(1)} r + \left(q_2 \pi_1 + a^{(0)} \pi_2 - r W \right) f \\ &= q a^{(1)} r + f \tilde{\pi}, \end{aligned}$$

where $\tilde{\pi}(x)$ is some polynomial of degree less than $2n$. But then necessarily $a^{(1)}(x)$ divides $\tilde{\pi}(x)$ and we have

$$t a^{(0)} = q r + f \pi, \qquad \deg \pi(x) < n,$$

which (by Theorem 2.1) is equivalent to

$$r^T L_{q/f} D_2 = t^T.$$

The rest of the proof follows by induction. ■

REMARK 3.1. On the basis of Remark 2.4 we are able to give some information about the rank of the product of LD-matrices. In general for a product of k matrices the following fact is known [11, p. 28]: If

$$M = M_1 M_2 \cdots M_k$$

then

$$\min_{i=1,2,\ldots,k} d(M_i) \leqslant d(M) \leqslant \sum_{I=1}^{k} d(M_i),$$

where $d(M)$ denotes the defect of the matrix M, that is,

$$d(M) = n - r(M).$$

Now we can characterize the case when the defect of the product of LD-matrices $L_{q_i/f}D_i$, $i = 1,2,\ldots,k$, attains the maximum value possible, that is,

$$d\left(L_{q/f}D_k\right) = \sum_{i=1}^{k} d\left(L_{q_i/f}D_i\right).$$

Suppose $k = 2$. The defect of the matrix

$$M^T = (M_1 M_2)^T \qquad \left(\text{where} \quad M_i = L_{q_i/f}D_i\right)$$

is equal to $d(M_1) + d(M_2)$ if and only if the kernel of M_2^T is contained in the range of M_1^T. Using the result of Remark 2.4 this is equivalent to the condition that the product $(\bar{q}_1, \bar{f})(\bar{q}_2, \bar{f})$ divides $\bar{f}$, where $\bar{q}_i, \bar{f}$ are forms defined in Remark 2.4. [We consider $f(x), q_i(x)$ as polynomials of degree n.] For an arbitrary k we obtain by induction

$$d\left(L_{q/f}D_k\right) = \sum_{i=1}^{k} d\left(L_{q_i/f}D_i\right)$$

if and only if $\prod_{i=1}^{k}(\bar{q}_i, \bar{f})$ divides $\bar{f}$.

This result could also be obtained on the basis of the formula (3.3). In this way it is possible to derive that in general

$$d\left(L_{q/f}D_k\right) = \deg\left(\prod_{i=1}^{k} \bar{q}_i, \bar{f}\right).$$

THEOREM 3.2. *Let $f(x)$ be a nonzero polynomial,* $\deg f(x) \leqslant n$, *and let* $y^{(0)}, y^{(1)}, \dots, y^{(k)}$, $k \geqslant 1$, *be vectors with n distinct components each. Further, let* $w_j(x)$, $j = 1,2,\dots,k$, *be polynomials of degree less than* $2n$; *choose the matrices*

$$L_{w_j} \text{ in } \mathscr{L}\big(y^{(j-1)}, y^{(j)}\big), \qquad j = 1,2,\dots,k.$$

Suppose the polynomials $w_j(x)$ *are such that all the matrices* L_{w_j} *are compatible with* $f(x)$. *If the diagonals* D_j *are as in* (3.1), *then*

$$L_{w_1}D_1L_{w_2}D_2 \cdots L_{w_k}D_k = L_w D_k \in \mathscr{Ld}_f\big(y^{(0)}, y^{(k)}\big), \tag{3.4}$$

where $w(x)$ *is determined from the equations*

$$q_j + h_j a^{(j-1)}a^{(j)} = fw_j, \quad \deg q_j \leqslant n, \qquad j = 1,2,\dots,k, \tag{3.5}$$

$$h = (-1)^{k-1}h_1h_2 \cdots h_k a^{(1)}a^{(2)} \cdots a^{(k-1)} + f\pi, \qquad \deg h < \deg f, \tag{3.6}$$

$$wa^{(1)} \cdots a^{(k-1)} = \frac{ha^{(0)} \cdots a^{(k)} + q_1 \cdots q_k}{f} + W, \qquad \deg W \leqslant (k-1)n. \tag{3.7}$$

(*The polynomial* $h(x)a^{(0)}(x) \cdots a^{(k)}(x) + q_1(x) \cdots q_k(x)$ *is divisible by* $f(x)$.)

REMARK 3.2. In the case $\deg(f, a^{(0)}a^{(k)}) > 0$ the matrix L is not uniquely determined by $q(x)$. Moreover, if $\deg(f, a^{(1)}a^{(2)} \cdots a^{(k-1)}) > 0$, the polynomial $q(x)$ cannot be uniquely determined from (3.3).

REMARK 3.3. The polynomials $h_j(x)$ in (3.5) are quotients in the division of $f(x)w_j(x)$ by $a^{(j-1)}(x)a^{(j)}(x)$ [the remainders $q_j(x)$ are of degrees at most n owing to the compatibility of $f(x)$ and $L_{w_j} \in \mathscr{L}(y^{(j-1)}, y^{(j)})$]. Equation (3.6) determines $h(x)$ uniquely from $h_1(x), \dots, h_k(x)$. For this $h(x)$ there exists a unique $w(x)$ satisfying (3.7).

Proof. If all L_{w_j}, $j = 1,2,\dots,k$, have the H-property, then $L_{w_j} = L_{q_j/f}$, where $q_j(x)$ are defined by (3.5) (see Theorem 1.4). The equality (3.4) follows from (3.2); the equalities (3.6) and (3.7) follow from (3.3).

If some of the L_{w_j}'s have not the H-property, we use analogous limit considerations to those in the second part of the proof of Theorem 1.3. ∎

4. INVERSES OF LOEWNER MATRICES

Considering the product $M = M_1 M_2$, where $M_1 \in \mathscr{L}\!\mathscr{d}_f(y, z)$, $M_2 \in \mathscr{L}\!\mathscr{d}_f(z, y)$, we have $M \in \mathscr{L}\!\mathscr{d}_f(y, y)$. We know that in the case $(f, a) = 1$ the identity matrix belongs to $\mathscr{L}\!\mathscr{d}_f(y, y)$, and now we shall find for any nonsingular matrix $M_1 \in \mathscr{L}\!\mathscr{d}_f(y, z)$ an $M_2 \in \mathscr{L}\!\mathscr{d}_f(z, y)$ such that $M_1 M_2 = I$.

Consequently, we find the inverse of an arbitrary nonsingular Loewner matrix, which generalizes, in a sense, the result of [12], where only the nonsymmetric case was considered.

We know that

$$L_{a/f}\operatorname{diag}\left(\frac{f(y_i)}{a_i(y_i)}\right) \in \mathscr{L}\!\mathscr{d}_f(y, y)$$

is the identity matrix if $(f, a) = 1$. But

$$L_{q_1/f} D_1 L_{q_2/f} D_2 = L_{a/f} D_2$$

—where $(L_{q_1/f} \in \mathscr{L}(y, z)$, $L_{q_2/f} \in \mathscr{L}(z, y)$,

$$D_1 = \operatorname{diag}\left(\frac{f(z_i)}{b_i(y_i)}\right), \qquad D_2 = \operatorname{diag}\left(\frac{f(y_i)}{a_i(y_i)}\right) \tag{4.1}$$

—if and only if

$$ab = q_1 q_2 + fW \tag{4.2}$$

for some $W(x)$. It is evident that for all $q_1(x)$, $(f, q_1) = 1$, there is a unique $q_2(x)$ of degree at most n satisfying (4.2) [up to a constant multiple of $f(x)$]. Thus we have:

THEOREM 4.1. *Let $f(x)$ be a nonzero polynomial of degree at most n, y, z vectors (each with n distinct components) such that $(f, ab) = 1$. Let*

$$M_1 \in \mathscr{L}\!\mathscr{d}_f(y, z)$$

be an arbitrary nonsingular LD-matrix,

$$M_1 = L_{q_1/f} D_1$$

(*every nonsingular Loewner matrix possesses some interpolating function by Theorem* 1.5). *Then*

$$M_1^{-1} \in \mathscr{L}\!\mathscr{A}_f(z, y), \qquad M_1^{-1} = M_2 = L_{q_2/f} D_2,$$

where

$$q_1 q_2 = ab - fW, \qquad \deg q_2 \leqslant n,$$

for some polynomial $W(x)$. (D_1, D_2 *have the form* (4.1).)

As a corollary we have (see [12, Theorem 2.1]):

THEOREM 4.2. *Let* $L_1 \in \mathscr{L}(y, z)$ *be a nonsingular Loewner matrix* (*we point out that the symbol* $\mathscr{L}(y, z)$ *is used for both symmetric and nonsymmetric Loewner matrices*). *Then the matrix* L_1^{-1} *can be written in the form*

$$L_1^{-1} = D_1 L_2 D_2 \tag{4.3}$$

where D_1, D_2 *are diagonal matrices and* L_2 *is a Loewner matrix belonging to* $\mathscr{L}(z, y)$. *More exactly, if*

$$L_1 = L_{q_1/f} \qquad (\deg q_1(x), \deg f(x) \leqslant n)$$

(*and such polynomials exist*), *then we can write*

$$L_2 = L_{q_2/f},$$

where $q_2(x)$ ($\deg q_2(x) \leqslant n$) *is such that*

$$q_1 q_2 = ab - fW$$

for some $W(x)$, *and* D_1, D_2 *have the form* (4.1).

(Reference [12] describes all possible forms (4.3).)

5. THE GROUP STRUCTURE OF *LD*-CLASSES IN THE SYMMETRIC CASE

If we take $y^{(0)} = y^{(1)} = \cdots = y^{(k)} = y$ in Theorem 3.2, then, considering Remark 2.4, the class $\mathscr{Ld}_f(y,y)$ becomes an algebraic structure with an associative and commutative operation possessing the unit element. Denoting by $\mathscr{Ld}_f^{(n)}(y,y)$ the subclass of all nonsingular matrices from $\mathscr{Ld}_f(y,y)$, we obtain an Abelian group.

THEOREM 5.1. *Let $f(x)$ be a nonzero polynomial of degree at most n, and y a vector with n distinct components. Then the class*

$$\mathscr{Ld}_f(y,y)$$

is an algebraic structure with the operation of matrix multiplication which is associative and commutative and possesses a unit element in $\mathscr{Ld}_f(y,y)$.

If $(f,a)=1$, then the subclass

$$\mathscr{Ld}_f^{(n)}(y,y)$$

of all nonsingular matrices from $\mathscr{Ld}_f(y,y)$ is an Abelian group.

Let us show that the identity element in $\mathscr{Ld}_f(y,y)$ need not be the identity matrix:

EXAMPLE 5.1. Let $n=3$, $y=(-1,0,1)^T$, and $f(x)=x(x^2+1)$. Then

$$\operatorname{diag}\left(\frac{f(y_i)}{a_i(y_i)}\right)^2_{i=0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$\frac{a(x)}{f(x)} = \frac{x^2-1}{x^2+1}, \quad \left(\frac{a(x)}{f(x)}\right)' = \frac{4x}{(x^2+1)^2},$$

we have

$$L_{a/f} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$L_{a/f}\operatorname{diag}\left(\frac{f(y_i)}{a_i(y_i)}\right)=\begin{pmatrix}1&0&0\\1&0&1\\0&0&1\end{pmatrix},$$

which is the identity element in $\mathscr{L}_f(y,y)$ but is not equal to the identity matrix. The reason is that

$$(f,a)=x.$$

REMARK 5.1. If $f(x)$ is the polynomial from Theorem 5.1, we define the vector space P_f as follows: Let P_n be the space of polynomials of degree at most n, and define an equivalency $\sim$ on P_n:

$$p_1(x)\sim p_2(x)\quad\text{if}\quad p_1(x)-p_2(x)=cf(x)$$

for some constant c. Now P_f is the factor space with respect to $\sim$. [In the case $\deg f(x)=n$ it is the usual space of polynomials modulo $f(x)$.] We shall denote by q^* the equivalence class containing the polynomial $q(x)$.

Now we can construct an isomorphism $\mathscr{K}$ between $\mathscr{L}_f^{(H)}(y,y)$ and P_f:

$$\mathscr{K}(M)=q^*\quad\text{if}\quad M=L_{q/f}D.$$

The definition makes sense, since $L_c=0$ for any constant c. This isomorphism yields the following representation of the algebraic structure of $\mathscr{L}^{(H)}(y,y)$: The multiplicative operation is represented in P_f by

$$q_1^*q_2^*=q^*\quad\text{if}\quad qa=q_1q_2+fW$$

for some $W(x)$ of degree at most n. The unit element is a^*.

In the case $(f,a)=1$ we have

$$\max(\deg q_1(x),\deg f(x))=n$$

for all $M_1\in\mathscr{L}_f^{(n)}(y,y)$ [where $q_1^*=\mathscr{K}(M_1)$], and there is a unique q_2^* in P_f defined by

$$a^2=q_1q_2+fW,\qquad \deg W\leqslant n.$$

This q_2^* is the inverse element to q_1^*.

REMARK 5.2. Choosing $y^{(0)} = y^{(k)}$ but in general not necessarily $y^{(0)} = y^{(1)} = y^{(2)} = \cdots = y^{(k-1)}$ in Theorem 3.2, we can perform any cyclic rearrangement of LD-matrices in the product

$$L_{w_1} D_1 L_{w_2} D_2 \cdots L_{w_k} D_k$$

and obtain

$$L_{w_{i+1}} D_{i+1} L_{w_{i+2}} D_{i+2} \cdots L_{w_k} D_k L_{w_1} D_1 \cdots L_{w_i} D_i = L_w D_i,$$

where the resulting matrix L_w has always the same interpolating polynomial $w(x)$ but belongs to various classes

$$\mathscr{LD}_f\big(y^{(i)}, y^{(i)}\big), \qquad i \in \{1,2,\dots,k\}.$$

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REFERENCES

1 V. Belevitch, Interpolation matrices, *Philips Res. Rep.* 25:337–369 (1970).
2 W. F. Donoghue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Springer, New York, 1974.
3 M. Fiedler, Hankel and Loewner matrices, *Linear Algebra Appl.* 58:75–95 (1984).
4 M. Fiedler, Polynomials and Hankel matrices, *Linear Algebra Appl.* 66:235–248 (1985).
5 M. Fiedler, Quasidirect decompositions of Hankel and Toeplitz matrices, *Linear Algebra Appl.* 61:155–174 (1984).
6 M. Fiedler and V. Pták, Bézoutians and intertwining matrices, *Linear Algebra Appl.* 86:43–51 (1987).
7 P. A. Fuhrmann, On the partial realization problem and the recursive inversion of Hankel and Toeplitz matrices, in *Linear Algebra and its Role in System Theory*, Contemp. Math., Amer. Math. Soc., 1985.
8 G. Heinig and K. Rost, *Algebraic Methods for Toeplitz-like Matrices and Operators*, Birkhäuser, Basel, 1984.
9 M. G. Krein and M. A. Naimark, The method of symmetric and Hermitian forms in the theory of the separation of the roots of algebraic equations, *Linear and Multilinear Algebra* 10:256–308 (1981).
10 K. Loewner, Über monotone Matrixfunktionen, *Math. Z.* 38:177–216 (1934).
11 M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston, 1964.
12 Z. Vavřín, Inverses of Loewner matrices, *Linear Algebra Appl.* 63:227–236 (1984).
13 Z. Vavřín, Loewner Matrices and the Generalized Cauchy Interpolation Problem (in Czech), Thesis, Charles Univ., Prague, 1983.

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